New Calabi-Yau manifolds with small Hodge numbers

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Outline

Calabi-Yau manifolds

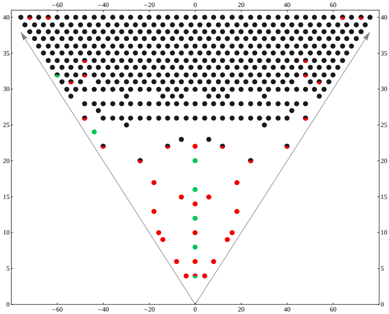
- For this talk, a Calabi-Yau manifold is a compact Kähler 3-fold with trivial first Chern class.
- This is enough to determine all except two Hodge numbers. The Hodge diamond is

$$
\begin{array}{ccccccc}
 & & & h^{00} & & & \\
 & & h^{10} & & h^{01} & & \\
 & h^{20} & & h^{11} & & h^{02} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{13} & & h^{22} & & h^{31} & \\
 & & h^{23} & & h^{32} & & \\
 & & & h^{33} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{21} & & h^{21} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

- Thus the Euler number is $\chi = 2(h^{11} - h^{21})$.

Triadophilia I

In a recent paper[1], it was observed that the bottom of the Calabi-Yau 'landscape' is relatively sparsely populated.



- The Kreuzer–Skarke list, CICY's, toric CICY's, and toric conifolds, with their mirrors.
- The Gross–Popescu, Rødlandand, Tonoli , Borisov-Hua and Hua manifolds.
- Previously known quotients by freely acting groups and their mirrors.
- Divided dots denote overlays.

[1] Candelas et al, *Triadophilia: A Special Corner in the Landscape*, **Adv.Theor.Math.Phys.12:2,2008**, arXiv:0706.3134

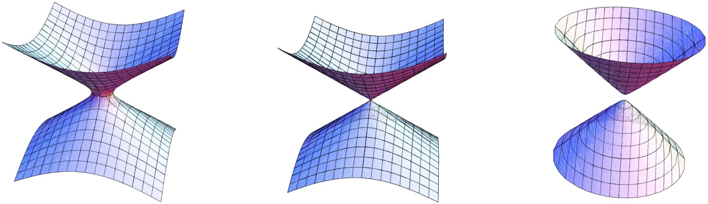
Triadophilia II

A number of other observations were made in the paper:

- There are at least two phenomenologically promising string models on manifolds in the ‘tip’, despite the scarcity of such manifolds.
- Almost all the manifolds known with small Hodge numbers have non-trivial fundamental group.

Conifold transitions

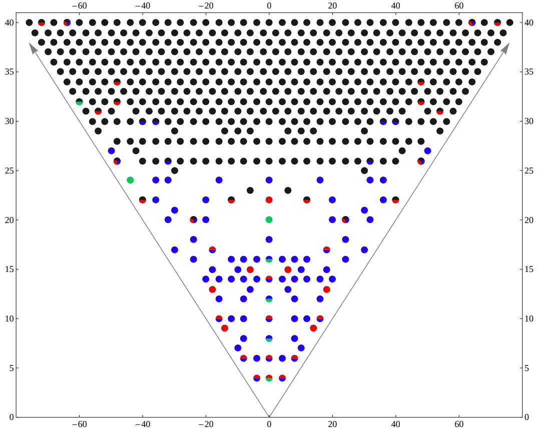
- A smooth manifold can be deformed until nodes develop, and these nodes resolved to yield another manifold with the same fundamental group. This is called a "conifold transition".



- Therefore perhaps we can find new manifolds in the tip via conifold transitions from those which are already known.
- This corresponds to finding a conifold of the covering space which respects the symmetry.

Results

We can plot the distribution of Hodge numbers in the tip including the new manifolds we have found

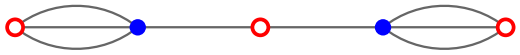
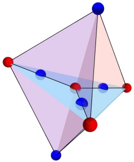


CICY's

- There are many constructions of Calabi-Yau manifolds; we need only the simplest, the Complete Intersection Calabi-Yau manifolds (CICY's).
- These are the common vanishing locus of some set of polynomials in a product of complex projective spaces.

Configurations and diagrams

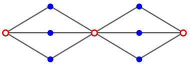
Two examples

Configuration matrix	Diagram
Ambient spaces $\left\{\begin{array}{c}\mathbb{P}^1\\ \mathbb{P}^2\\ \mathbb{P}^2\end{array}\right.\!\!\begin{bmatrix}1 & 1\\ 3 & 0\\ 0 & 3\end{bmatrix}^{19,19}$	
$\begin{array}{c}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{array}\!\!\begin{bmatrix}1 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0 & 0\end{bmatrix}^{4,40}$	
Columns represent polynomials, and entries the degree in the variables of each space.	Open circles are spaces and dots are polynomials. The number of lines represents the degree.

Quotient manifolds
An example

- Consider the configuration

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \end{array}\left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]^{3,39}$$



- Take coordinates u_i on the first $\mathbb{P}^2$, v_i on the second, and (x_i, y_j) on $\mathbb{P}^5$. Then we can define an action of $\mathbb{Z}_3$:

$$S: \ u_i \longrightarrow u_{i+1}, \ v_i \longrightarrow v_{i+1}, \ (x_i, y_j) \longrightarrow (x_{i+1}, y_{j+1})$$

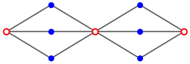
- We want our manifold invariant, so choose our polynomials such that

$$S: \ p_i \longrightarrow p_{i+1}, \ q_i \longrightarrow q_{i+1}$$

Quotient manifolds

An example

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{3,39}$$



- The appropriate polynomials are (here $i \in \mathbb{Z}_3$)

$$p_i = \sum_{jk} (A_{jk}\, x_{i+j} + B_{jk}\, y_{i+j})\, u_{i+k}$$

$$q_i = \sum_{jk} (C_{jk}\, x_{i+j} + D_{jk}\, y_{i+j})\, v_{i+k}$$

- The group acts without fixed points, so we obtain a smooth quotient manifold.

The conifold

- Consider the CICY given by the configuration

$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

  and take coordinates $\{x\}$ on the first $\mathbb{P}^2$ and $\{y\}$ on the second.

- We will look at a degenerate form of the defining equation:

$$U(x)V(y) - W(x)Z(y) = 0$$

  where U, V, W, Z are cubics. The variety obviously has (nodal) singularities at points where $U = V = W = Z = 0$.

- Such singular varieties are known to physicists as "conifolds", since the neighbourhood of a node is a cone over $S^3 \times S^2$.

The conifold

Deforming

- The conifold can be smoothed by a small change to the defining equation:

$$U(x)V(y) - W(x)Z(y) + \epsilon K(x, y) = 0$$

  This is called a *deformation*.
- The conifold should be thought of as a limit point (as $\epsilon \to 0$) of the moduli space of smooth manifolds.

The conifold

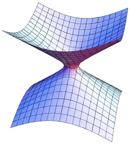
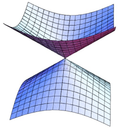
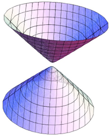
Resolving

- Now consider the pair of equations given by

$$\begin{pmatrix} U(x) & Z(y) \\ W(x) & V(y) \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \left(\text{or } \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \right)$$

- These have solutions iff $UV - WZ = 0$, which is our singular equation again. Now though, when $U = V = W = Z = 0$, we have a whole $\mathbb{P}^1$ of solutions parametrised by $[t_0 : t_1]$.
- The resulting variety, which is smooth, is called a *resolution* of the conifold.

Conifold transitions

Geometry			
Equation	$U(x)V(y) - W(x)Z(y)$ $+ \epsilon K(x,y) = 0$	$U(x)V(y) - W(x)Z(y) = 0$	$\begin{pmatrix} U(x) & Z(y) \\ W(x) & V(y) \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Family	$\begin{matrix}\mathbb{P}^2 \\ \mathbb{P}^2\end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ Smooth	$\begin{matrix}\mathbb{P}^2 \\ \mathbb{P}^2\end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ Singular	$\begin{matrix}\mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2\end{matrix} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}$ Smooth

Splitting a configuration

- More generally we can introduce a $\mathbb{P}^n$ to split a polynomial of total degree at least $n+1$:

$$\mathcal{P}[M, \mathbf{c}] \;\rightarrow\; \begin{array}{c} \mathbb{P}^n \\ \mathcal{P} \end{array} \begin{bmatrix} \mathbf{0} & 1 & 1 & \cdots & 1 \\ M & \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_{n+1} \end{bmatrix}$$

  where $\sum_i \mathbf{c}_i = \mathbf{c}$.

- For example,

$$\begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^5 \end{array} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{array}{c} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^5 \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Conifold transitions in string theory

- Geometrically, a conifold is singular, but in string theory, the physics is still sensible.
- So it is possible for spacetime to undergo a conifold transition!
- Seemingly distinct string vacua are therefore actually connected.

A quotient of the quintic

It has been known for a long time that the quintic $\mathbb{P}^4[5]$ admits a free action by $\mathbb{Z}_5$. Let's see how this works:

- Take the following special case of the defining equation

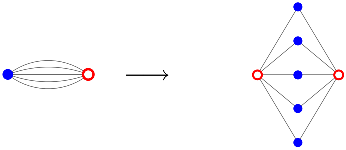
$$(x_0)^5 + (x_1)^5 + (x_2)^5 + (x_3)^5 + (x_4)^5 + \alpha\, x_0 x_1 x_2 x_3 x_4 = 0$$

- There is a free $\mathbb{Z}_5$ action given by $x_i \rightarrow x_{i+1}$.

Splitting the quintic

We want to split the quintic in a way which might preserve this $\mathbb{Z}_5$ symmetry. There is a natural candidate:

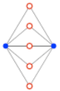
$$\mathbb{P}^4[5]^{1,101} \quad \longrightarrow \quad \begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^4\end{matrix}\begin{bmatrix}1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1\end{bmatrix}^{2,52}$$



The $\mathbb{Z}_5$ web

Continuing in a similar fashion, we obtain the following “web” of $\mathbb{Z}_5$ quotients

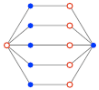
$\mathbb{P}^4[5]^{1,101}_{\{\mathbb{Z}_5\times\mathbb{Z}_5\}}$



$$\begin{array}{c} \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1 \end{array}\left[\begin{array}{cc} 1&1\\ 1&1\\ 1&1\\ 1&1\\ 1&1 \end{array}\right]^{5,45}_{\{\mathbb{Z}_5\times\mathbb{Z}_2\times\mathbb{Z}_2\}}$$

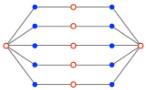
$\downarrow$ $\downarrow$

$$\begin{array}{c} \mathbb{P}^4\\ \mathbb{P}^4 \end{array}\left[\begin{array}{ccccc} 1&1&1&1&1\\ 1&1&1&1&1 \end{array}\right]^{2,52}_{\{\mathbb{Z}_5\times\mathbb{Z}_2\}}$$



$$\begin{array}{c} \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^4 \end{array}\left[\begin{array}{cccccc} 1&1&0&0&0&0\\ 1&0&1&0&0&0\\ 1&0&0&1&0&0\\ 1&0&0&0&1&0\\ 1&0&0&0&0&1\\ 0&1&1&1&1&1 \end{array}\right]^{6,36}_{\{\mathbb{Z}_5\}}$$

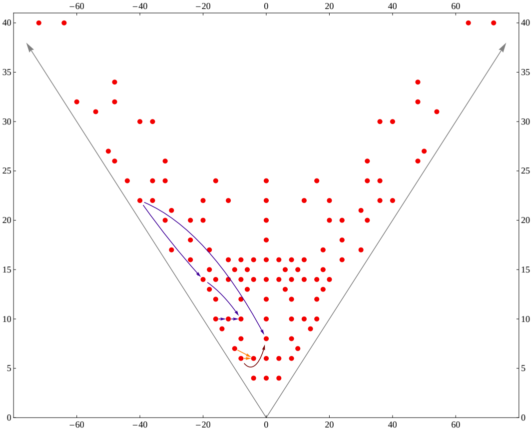
$\searrow$ $\swarrow$



$$\begin{array}{c} \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^1\\ \mathbb{P}^4\\ \mathbb{P}^4 \end{array}\left[\begin{array}{cccccccccc} 1&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1\\ 1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&1&1&1&1&1 \end{array}\right]^{7,27}_{\{\mathbb{Z}_5\times\mathbb{Z}_2\}}$$

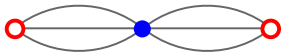
The $\mathbb{Z}_5$ web

We can also plot the conifold transitions on the Hodge numbers diagram

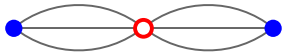


The following two manifolds each admit a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ action.

$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

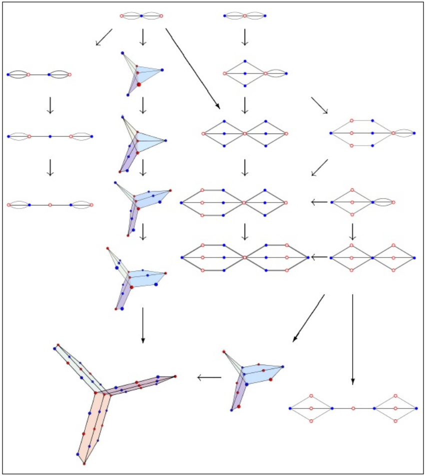


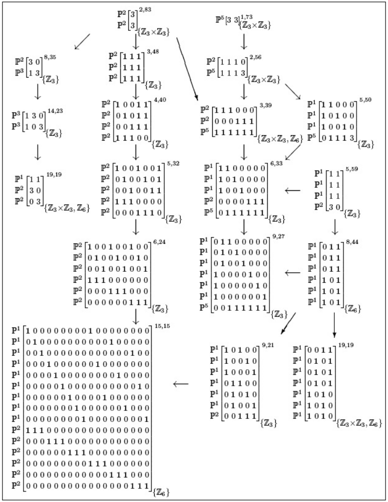
$$\mathbb{P}^5 \begin{bmatrix} 3 & 3 \end{bmatrix}$$



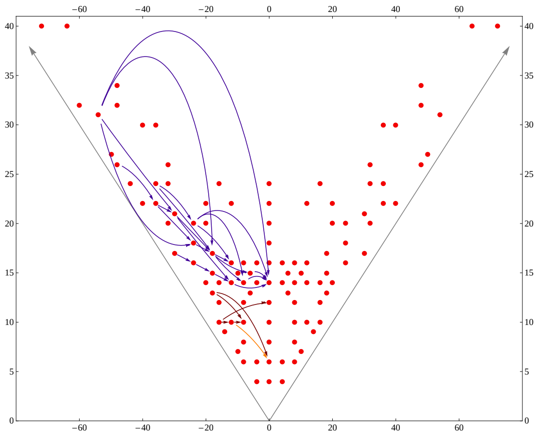
We can split each of these repeatedly to obtain the following web of manifolds admitting free $\mathbb{Z}_3$ actions.

The $\mathbb{Z}_3$ web





The $\mathbb{Z}_3$ web



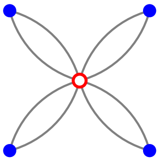
The quaternion group

- Finally we found free actions by the order 8 quaternion group:

$$\{1, i, j, k, -1, -i, -j, -k\}$$

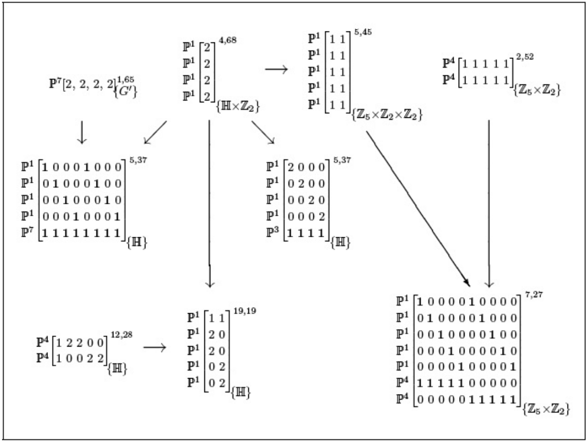
- The starting point is the following, already known to admit an $\mathbb{H}$ action

$$\mathbb{P}^7[2\ 2\ 2\ 2]^{1,65}$$



The $\mathbb{H}$ web

By splitting the above manifold we obtain a number of new examples:



Matrix transposition

- For each matrix appearing in one of our webs, the transpose also appears.
- This is puzzling:
  - Transposition has no obvious geometrical meaning.
  - Even if two matrices give the same manifold, their transposes can give different manifolds.

Matrix transposition

Example: the following are equivalent

$$\begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\end{matrix}\begin{bmatrix}\mathbf{1}&0&\mathbf{1}\\\mathbf{1}&0&\mathbf{1}\\\mathbf{1}&0&\mathbf{1}\\0&\mathbf{1}&\mathbf{1}\\0&\mathbf{1}&\mathbf{1}\\0&\mathbf{1}&\mathbf{1}\end{bmatrix}^{8,44} \cong \begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^2\\\mathbb{P}^2\end{matrix}\begin{bmatrix}\mathbf{1}&0&0&0&0&0&\mathbf{1}\\0&\mathbf{1}&0&0&0&0&\mathbf{1}\\0&0&\mathbf{1}&0&0&0&\mathbf{1}\\0&0&0&\mathbf{1}&0&0&\mathbf{1}\\0&0&0&0&\mathbf{1}&0&\mathbf{1}\\0&0&0&0&0&\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&0&0\\0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&0\end{bmatrix}^{8,44}$$

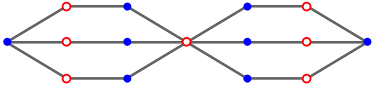
but the transposes have different Hodge numbers

$$\begin{matrix}\mathbb{P}^2\\\mathbb{P}^2\\\mathbb{P}^5\end{matrix}\begin{bmatrix}\mathbf{1}&\mathbf{1}&\mathbf{1}&0&0&0\\0&0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}\\\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}^{3,39} \quad \begin{matrix}\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^1\\\mathbb{P}^5\end{matrix}\begin{bmatrix}0&\mathbf{1}&\mathbf{1}&0&0&0&0&0\\0&\mathbf{1}&0&\mathbf{1}&0&0&0&0\\0&\mathbf{1}&0&0&\mathbf{1}&0&0&0\\\mathbf{1}&0&0&0&0&\mathbf{1}&0&0\\\mathbf{1}&0&0&0&0&0&\mathbf{1}&0\\\mathbf{1}&0&0&0&0&0&0&\mathbf{1}\\0&0&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}^{9,27}$$

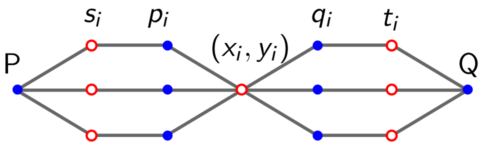
A new $\chi = -6$ manifold

The following occurs in the $\mathbb{Z}_3$ web:

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^5 \end{array} \begin{bmatrix} 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix}^{9,27}_{-36}$$



Label coordinates and polynomials as follows (with $i \in \mathbb{Z}_3$)



We can then define a $\mathbb{Z}_3 \times \mathbb{Z}_2$ action with generators

$$S : \quad (x_i, y_i, t_i, s_i) \to (x_{i+1}, y_{i+1}, t_{i+1}, s_{i+1}) \; ; \quad p_i \to p_{i+1}, q_i \to q_{i+1}$$
$$U : \quad x \leftrightarrow y, t \leftrightarrow s \; ; \quad p_i \leftrightarrow q_i, P \leftrightarrow Q$$

A new $\chi = -6$ manifold

- S acts without fixed points, but the fixed points of U correspond to two copies of

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

- These are one-dimensional CICY's i.e. tori.

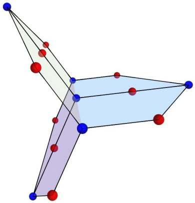
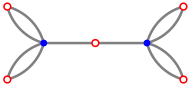
- Thus they have Euler number 0, and we can resolved them without changing the Euler number.

Euler number 0 manifolds

- The U.Penn. group has constructed a promising heterotic string model on the following manifold

$$\left(\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{bmatrix}_{/\mathbb{Z}_3 \times \mathbb{Z}_3} \right)^{3,3}$$

- We now have two more manifolds with Hodge numbers $3, 3$.

Manifold		
Group	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{H}$

These quotient manifolds both have Hodge numbers $(h^{11}, h^{21}) = (3, 3)$.

Summary

- We have found a number of new multiply-connected manifolds with small Hodge numbers.
- At least two of these manifolds resemble existing manifolds on which promising string models have been contructed.
- Scope for new model building, or general study of multiply-connected manifolds in string theory.